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A Counterexample on Global Chebyshev Approximation

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Let S be a compact Hausdorff space. The space of continuous, real-valued functions on S is denoted by $C(S)$. This paper gives a counterexample for the problem given by Franchetti and Cheney in (Franchetti and Cheney, *in* "Approximation Theory and Its Applications," Academic Press, New York, 1981; Cheney, *in* "Approximation Theory IV," Academic Press, New York, 1983). The problem is as follows: If Y is a proximinal set in $C(S)$, is it also proximinal in $C(S \times T)$ for arbitrary compact T? © 1987 Academic Press, Inc.

1. INTRODUCTION

In a normed linear space X, the distance from a point x to a subset Y $(\neq \varnothing)$ is defined by

$$\operatorname{dist}(x, Y) = \inf\{\|x - y\|: y \in Y\}.$$

If an element y in Y satisfies $\|x - y\| = \operatorname{dist}(x, Y)$, then y is called a best approximation of x in Y. If each $x \in X$ has at least one best approximation in Y, then Y is termed proximinal.

If A and Y are subsets of a Banach space X, then the Chebyshev radius of A relative to Y is

$$r_Y(A) = \inf_{y \in Y} \sup_{a \in A} \|a - y\|.$$

The Chebyshev center of A relative to Y is

$$E_Y(A) = \{y \in Y: \sup_{a \in A} \|a - y\| = r_Y(A)\}.$$

If $Y = X$, $r_Y(A)$ and $E_Y(A)$ are written as $r(A)$ and $E(A)$ and termed the Chebyshev radius and the Chebyshev center of A, respectively.

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An interval in $C(S)$ is a set of the form

$$[a, b] = \{x \in C(S): a \leq x \leq b\}$$

where a and b are assumed to be bounded functions on S. An interval is of type II if $a, b \in C(S)$ and there exists a point s_0 such that $a(s_0) = b(s_0)$. We denote

$$(a \vee b) = \max(a(s), b(s))$$

$$(a \wedge b) = \min(a(s), b(s)),$$

where a, b are arbitrary elements in $C(S)$.

Franchetti and Cheney [1] showed that the Chebyshev center of any bounded subset of $C(S)$ is non-empty. Because of this, we want to know whether $E_Y(A)$ is also non-empty, where Y is a proximinal subset in $C(S)$ and A is a bounded compact subset in $C(S)$. By [3], we can see that $E_Y(A)$ is non-empty if and only if Y is proximinal in $C(S \times T)$ for each compact Hausdorff space T.

Franchetti and Cheney [3] show the following theorem:

THEOREM 1.1. *Let S be a compact Hausdorff space and Y be a proximinal subset of $C(S)$. In order that Y be a proximinal subset of $C(S \times T)$ for every compact Hausdorff space T, it is necessary and sufficient that* $\operatorname{dist}(x, Y)$ *attain its infimum on every interval X of type* II.

Thus, if we can find a proximinal set Y such that $\operatorname{dist}(x, Y)$ does not attain its infimum for some $X = [u, v]$, then we will have finished our discussion.

2. Some Lemmas

In order to construct the counterexample, we give some preliminary lemmas.

LEMMA 2.1. *Let $u \in C(S)$, $v \in C(S)$, and $u \leq v$. Let G be a proximinal set in $C(S)$. Put $d(x) = \operatorname{dist}(x, G)$ and $d = \inf\{d(x): u \leq x \leq v\}$. In order that this infimum be attained by an x in $[u, v]$ it is necessary and sufficient that $G \cap [u - d, v + d]$ be non-empty.*

Proof. Let $u \leq x \leq v$ and $\operatorname{dist}(x, G) = d$. There is a $g \in G$ such that $\|x - g\| = \operatorname{dist}(x, G)$. Then $-d \leq g - x \leq d$, and hence

$$u - d \leq x - d \leq g \leq x + d \leq v + d.$$

Hence $g \in G \cap [u - d, v + d]$.

Now suppose that $G \cap [u - d, v + d]$ is non-empty. Let g be an element of this set. Define $x=(g \vee u) \wedge v$. Clearly $x \leqslant v$. For each s, either $x(s)=v(s)$ or $x(s)<v(s)$. In the first case $x(s) \geqslant u(s)$ since $u(s) \leqslant v(s)$. In the second case $x(s)=(g \vee u)(s) \geqslant u(s)$. Hence $x \geqslant u$. Now

$$\begin{aligned}(x-g)(s) &= [(g \vee u)-g](s) \wedge (v-g)(s) \\ &= [(0 \vee (u-g)) \wedge (v-g)](s) \leqslant 0 \vee (u-g)(s) \leqslant d.\end{aligned}$$

Also $0 \vee (u-g)(s) \geqslant 0 \geqslant -d$, and $(v-g)(s) \geqslant -d$. Therefore $[(0 \vee (u-g)) \wedge (v-g)](s) \geqslant -d$ and $(x-g)(s) \geqslant -d$. This proves that $-d \leqslant x-g \leqslant d$ and that $\|x-g\| \leqslant d$. Thus the infimum defining $d(x)$ is attained at x.

LEMMA 2.2. *For any constant c, the set $V=\{f \in C(S)\colon \min_{s \in S} f(s) \leqslant c\}$ is proximinal in $C(S)$.*

Proof. Assume $t \in C(S) \backslash V$. Let $\bar{t}(s)=t(s)-\min_s t(s)+c$. Obviously, $\bar{t} \in V$ and $\bar{t}$ is a best approximation of t in V. In fact, for any $k \in V$, suppose $k(s_0)=\min_s k(s)$. Then

$$\|t-k\| \geqslant t(s_0)-k(s_0) \geqslant \min_s t(s)-c=\|\bar{t}-t\|.$$

Indeed,

$$\|\bar{t}-t\| \leqslant \|t-k\| \qquad \text{for all} \quad k \in K.$$

Therefore the lemma is true.

Let $S=[0,1]$, $A=\{\frac{1}{8}\}$, $V=\{g \in C(S)\colon \min_s g(s) \leqslant -\frac{1}{8}\}$, $B=\{g_k : k=1,2,\ldots\}$, where

$$g_k(s)=\begin{cases} -(k+1)s+1+\dfrac{1}{k}, & 0 \leqslant s \leqslant \dfrac{1}{k}; \\ 0, & \dfrac{1}{k} \leqslant s \leqslant 1 \end{cases} \qquad \text{(see Fig. 1)}.$$

Put $G=A \cup V \cup B$.

LEMMA 2.3. *The G defined above is a proximinal subset in $C(S)$.*

Proof. We divide our proof into the following cases:

1. $f=0$.
2. There exists s_0 such that $f(s_0) \leqslant -\frac{1}{8}$.
3. $0 \not\equiv f>-\frac{1}{8}$ and there exists s_0 such that $f(s_0)=-\|f\|$.
4. $0 \not\equiv f>-\frac{1}{8}$ and there exists s_0 such that $f(s_0)=\|f\| \leqslant \frac{1}{2}$.
5. $0 \not\equiv f>-\frac{1}{8}$ and there exists s_0 such that $f(s_0)=\|f\|>\frac{1}{2}$.

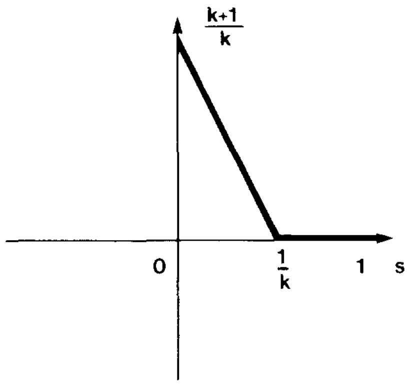

FIGURE 1

In the first case, obviously $g(s)=\frac{1}{8}$ is a best approximation of f in G. In the second case, it is trivial, for f belongs to G. In case 3, since $f(s_0)=-\|f\|>-\frac{1}{8}$, we have $\|f\|<\frac{1}{8}$. Therefore $|g_k(1)-f(1)|\geqslant\frac{7}{8}$, for every $k\geqslant 1$. Hence the best approximation of f in G must be found in $A\cup V$. In the fourth case, because of $\|f\|\leqslant\frac{1}{2}$, we thus have $|g_k(1)-f(1)>\frac{1}{2}$ for every $k\geqslant 1$. Hence each g_k is *not* a best approximation of f, for $|f(s)-\frac{1}{8}|\leqslant \frac{1}{8}+\frac{1}{8}=\frac{1}{4}<\frac{1}{2}$ when $f(s)\leqslant 0$ (Noticing $f(s)>-\frac{1}{8}$) $|f(s)-\frac{1}{8}|\leqslant\frac{1}{2}-\frac{1}{8}=\frac{3}{8}<\frac{1}{2}$ when $f(s)>0$.

Therefore the best approximation of f can be found in $A\cup V$.

In case 5, since f is a continuous function and $f(s_0)>\frac{1}{2}$ there must exist a point $s_1>s_0$ such that $f(s_1)>\frac{1}{2}$. Hence there exists an integer N such that

$$g_k(s_1)=0 \qquad \text{for all} \quad k\geqslant N.$$

Therefore

$$\|f-g_k\|\geqslant f(s_1)>\tfrac{1}{2} \qquad \text{for all} \quad k\geqslant N.$$

But

$$|f(s)-\tfrac{1}{8}|\leqslant\tfrac{1}{8}+\tfrac{1}{8}=\tfrac{1}{4} \qquad \text{when} \quad f(s)\leqslant 0,$$

$$|f(s)-\tfrac{1}{8}|\leqslant f(s_0)-\tfrac{1}{8}<f(s_1) \qquad \text{when} \quad f(s\geqslant 0.$$

These imply that $\|f-\frac{1}{8}\|<\|g_k-f\|$ for all $k\geqslant N$. Thus we can find a best approximation in $A\cup V\cup\{g_k: k=1, 2,..., N-1\}$.

3. MAIN RESULT

Now we give a counterexample for the problem. Let $v(s)=1-s$, $u(s)=0$. (See Fig. 2.) Then $[u, v]$ is an interval of type II.

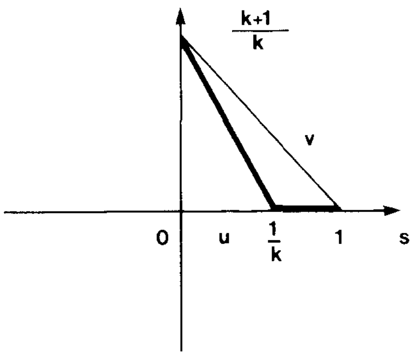

FIGURE 2

Since $g_k \in G$, we have $\text{dist}(G, [u, v]) \leqslant \text{dist}(g_k, [u, v])$ for all positive integers k. But we know $\lim_{k \to \infty} \text{dist}(g_k, [u, v]) = 0$, from which we have

$$\text{dist}(G, [u, v]) = 0.$$

According to the construction of G, there is no $g \in G$ such that $g \in [u, v]$.

By using Lemma 2.1 we know that $\text{dist}(G, [u, v])$ cannot attain its infimum. Since G is a proximinal subset in $C(S)$ (see Lemma 2.3), from Theorem 1.1 we can see that there must exist a compact Hausdorff space T such that G is not proximinal in $C(S \times T)$.

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